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The transformation of polynomial eigenfunctions of linear second-order difference operators: a special case of Meixner polynomials

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Abstract
As polynomial eigenfunctions of a linear second-order difference operator, a special case of the Meixner polynomials, $M_n^{(2,c)}(x+1)$, is transformed explicitly, into new complete sequences of non-classical orthogonal polynomials.

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1. Introduction

Consider the factorization chain

$$\begin{aligned} H_j(x) - \mu_j &= L_j(x)R_j(x) \\ H_{j+1}(x) - \mu_j &= R_j(x)L_j(x) \qquad j \in \mathcal{Z} \quad x \in \mathcal{C} \end{aligned} \tag{1}$$

where H_j is a second-order linear operator, and L_j and R_j are first-order linear operators. Such a factorization chain can be used to generate an 'exactly solvable' (in complete sequences of orthogonal polynomial solutions) second-order linear operator. For that, three kinds of method can be applied.

The first method consists in imposing on the factorization chain 'quasi-periodicity' conditions (see [27]). Such a method has been used to generate the Charlier, Kravchuk and Meixner polynomials (see [4]) as well as their q-versions (see [27]). It has also been used to generate a non-classical sequence of orthogonal polynomials related to the Hermite polynomials (see [7]).

The second method consists in imposing on the factorization chain another kind of self-similarity, a particular case of the so-called 'shape invariance' (see e.g. [15]). More precisely, one demands that the variable j in equation (1) acts not simply as an index but as a full independent variable. The method has been applied in [20] to generate the Charlier, Kravchuk, Meixner and Hahn polynomials. In addition, the method has been applied to the cases of hypergeometric polynomials on linear lattices (see [2]) and on q-nonlinear lattices (see [3]). It can also be applied to generate special cases of the so-called semiclassical or more generally Laguerre–Hahn polynomials [17, 18], as can be seen from [4,5]. Note that the shape-invariance method was used in [16] to generate a non-classical sequence of orthogonal polynomials related to the Jacobi polynomials.

The third method (in which we are interested here) consists in imposing on the factorization chain that two successive links belong to totally different families. In that case, an operator (Hamiltonian) in a given position in the chain is seen as a transformation of the operator lying in the preceding one. The method was used in [26] to modify the Hermite polynomials into non-classical sequences of orthogonal polynomials (including the above-cited ones from [7]). In this paper, the method is used to modify efficiently the special case of the Meixner polynomials, $M_n^{(2,c)}(x+1)$, as polynomial eigenfunctions of a linear second-order difference operator, into new complete sequences of non-classical orthogonal polynomials.

In the following section, we give a detailed discussion of the third method when applied to a general second-order linear difference operator and especially when applied to the second-order difference operator of hypergeometric type (see equation (15)). One will note for example the link between the method and the commonly used Christoffel and Geronimus transformations (see e.g. [10, 11, 28]). The third section, the main part of this paper, is devoted to the evoked explicit transformation of the special Meixner $M_n^{(2,c)}(x+1)$ polynomials. The last section is essentially a concluding one. An outlook is also given there. Finally, an appendix is given. It contains, for some instances of new polynomials, a collection in tables of a few important data such as intervals of orthogonality, weight functions and the first four polynomials.

2. Generalities

Let us first consider a general situation. The studied second-order difference operator H,

$$H(x) = u(x)\boldsymbol{E}_x + v(x) + w(x)\boldsymbol{E}_x^{-1} \tag{2}$$

$\boldsymbol{E}_x^i[h(x)] = h(x+i)$, $i \in \mathcal{Z}$, is, in the first place, supposed to act in a linear space of functions, say $\mathcal{L}$, and to admit a non-empty set of eigenfunctions. One can pick $(\Psi_{\hat{\alpha}}, \lambda_{\hat{\alpha}})$ as one of its pairs of eigenfunctions and corresponding eigenvalues. In that case, one can factorize H and then define $\tilde{H}$ as follows:

$$H - \lambda_{\hat{\alpha}} = L_{\hat{\alpha}} R_{\hat{\alpha}} \qquad \tilde{H} - \lambda_{\hat{\alpha}} = R_{\hat{\alpha}} L_{\hat{\alpha}} \tag{3}$$

where

$$\begin{aligned} R_{\hat{\alpha}} &= 1 + f_{\hat{\alpha}}(x)\boldsymbol{E}_x^{-1} & L_{\hat{\alpha}} &= u(x)\boldsymbol{E}_x + g_{\hat{\alpha}}(x) \\ f_{\hat{\alpha}}(x) &= -\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)} & g_{\hat{\alpha}}(x) &= -w(x)\frac{\Psi_{\hat{\alpha}}(x-1)}{\Psi_{\hat{\alpha}}(x)}. \end{aligned} \tag{4}$$

As it can be easily seen, the similarity reductions

$$\rho H \rho^{-1} \qquad \tilde{\rho}\tilde{H}\tilde{\rho}^{-1} \tag{5}$$

where

$$\frac{\rho^2(x+1)}{\rho^2(x)} = \frac{u(x)}{f_{\hat{\alpha}}(x+1)g_{\hat{\alpha}}(x+1)} = \frac{u(x)}{w(x+1)} \qquad \frac{\tilde{\rho}^2(x+1)}{\tilde{\rho}^2(x)} = \frac{u(x)}{f_{\hat{\alpha}}(x+1)g_{\hat{\alpha}}(x)} \tag{6}$$

allow us to transform H and $\tilde{H}$ into their *formal* symmetric form (i.e. like $A(x+1)\boldsymbol{E}_x + B(x) + A(x)\boldsymbol{E}_x^{-1}$). Denote by $\ell^2(a, b; \rho^2)$ the linear space of vectorial functions ($\psi(a)$,

$\psi(a+1), \ldots, \psi(b))$, in which is defined a discrete-weighted inner product

$$(\psi, \phi)_\rho = \sum_a^b \psi(x)\phi(x)\rho^2(x) \qquad -\infty < a < b \leqslant +\infty \tag{7}$$

(clearly, such a summation is performed on $a, a+1, \ldots, b$). A thus-defined $\ell^2(a, b; \rho^2)$ is well known to be a separable Hilbert space (see e.g. [1], for a general theory).

The similar space for $\tilde{\rho}^2$ will be denoted by $\tilde{\ell}^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$.

Letting $\{\Psi_\alpha, \lambda_\alpha\}$, $\alpha \in N_\alpha$ (a denumerable index set, not containing $\hat{\alpha}$), be a set of eigenfunctions and corresponding mutually different eigenvalues of H, one easily finds that the set $(\tilde{\Psi}_{\hat{\alpha}}, \lambda_{\hat{\alpha}})$, $\{\tilde{\Psi}_\alpha, \lambda_\alpha\}$, $\alpha \in N_\alpha$ where

$$\tilde{\Psi}_\alpha = R_{\hat{\alpha}}\Psi_\alpha \qquad L_{\hat{\alpha}}\tilde{\Psi}_{\hat{\alpha}} = 0 \tag{8}$$

is a resulting set of eigenfunctions and corresponding mutually different eigenvalues of $\tilde{H}$. Moreover, the completeness of $\tilde{\Psi}_{\hat{\alpha}}$, $\tilde{\Psi}_\alpha$, $\alpha \in N_\alpha$ in $\tilde{\ell}^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$ can be deduced from that of Ψ_α, $\alpha \in N_\alpha$ in $\ell^2(a, b; \rho^2)$ in the sense of the following proposition.

Proposition 2.1. *Supposing that:*

(c1) $L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}} \in \ell^2(a, +\infty; \rho^2)$, *and* $\{R_{\hat{\alpha}}\Theta_\alpha\} \in \tilde{\ell}^2(a+1, +\infty; \tilde{\rho}^2)$, *for* $\{\Theta_\alpha\}$ *and* $\tilde{\Theta}_{\hat{\alpha}}$ *belonging to the first and second space respectively.*
(c2) $\Theta_\alpha(x) = \tilde{\Theta}_{\hat{\alpha}}(x) = 0$, $x < a+1$.
(c3) $\alpha \in N_\alpha$, $\{\Theta_\alpha\}$ *is complete in* $\ell^2(a+1, +\infty; \rho^2)$.

Then:

(r1) The operators $L_{\hat{\alpha}}$ *and* $R_{\hat{\alpha}}$ *are* $(\rho^2, \tilde{\rho}^2)$*-mutually adjoint, in the sense that*

$$(\tilde{\Theta}_{\hat{\alpha}}, R_{\hat{\alpha}}\Theta_\alpha)_{\tilde{\rho}^2} = (L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}}, \Theta_\alpha)_{\rho^2}. \tag{9}$$

(r2) $\tilde{\Theta}^0_{\hat{\alpha}}$ *and* $\{R_{\hat{\alpha}}\Theta_\alpha\}$, $\alpha \in N_\alpha$, *are complete in* $\tilde{\ell}^2(a+1, +\infty; \tilde{\rho}^2)$, *where*

$$\tilde{\Theta}^0_{\hat{\alpha}}(x) = \begin{cases} \tilde{Y}_{\hat{\alpha}}(x) & x \geqslant a+1 \\ 0 & x < a+1 \end{cases}$$

$\tilde{Y}_{\hat{\alpha}}(x)$ *being defined by*

$$L_{\hat{\alpha}}\tilde{Y}_{\hat{\alpha}}(x) = 0. \tag{10}$$

Proof. Simple summation by parts gives

$$\begin{aligned}(L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}}, \Theta_\alpha)_{\rho^2} &= \sum_a^{+\infty} u(x)\tilde{\Theta}_{\hat{\alpha}}(x+1)\Theta_\alpha(x)\rho^2(x) + \sum_a^{+\infty} g_{\hat{\alpha}}(x)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_\alpha(x)\rho^2(x) \\ &= \sum_{a+1}^{+\infty} u(x-1)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_\alpha(x-1)\rho^2(x-1) + \sum_a^{+\infty} g_{\hat{\alpha}}(x)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_\alpha(x)\rho^2(x) \\ &= \sum_{a+1}^{+\infty} u(x-1)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_\alpha(x-1)\rho^2(x-1) + \sum_{a+1}^{+\infty} g_{\hat{\alpha}}(x)\tilde{\Theta}_{\hat{\alpha}}(x)\Theta_\alpha(x)\rho^2(x)\end{aligned} \tag{11}$$

where we used $(c2)$ for the last equality. On the other side

$$\begin{aligned}(\tilde{\Theta}_{\hat{\alpha}}, R_{\hat{\alpha}}\Theta_\alpha)_{\tilde{\rho}^2} &= \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x)[\Theta_\alpha(x) + f_{\hat{\alpha}}(x)\Theta_\alpha(x-1)]\tilde{\rho}^2(x) \\ &= \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x)\Theta_\alpha(x)\tilde{\rho}^2(x) + \sum_{a+1}^{+\infty} \tilde{\Theta}_{\hat{\alpha}}(x) f_{\hat{\alpha}}(x)\Theta_\alpha(x-1)\tilde{\rho}^2(x).\end{aligned} \tag{12}$$

Remarking that the equality of equations (11) and (12) requires the conditions set by the definitions in equation (6), this proves $(r1)$.

We remark next that, to prove the completeness of the system in $(r2)$ in $\tilde{\ell}^2(a+1, +\infty; \tilde{\rho}^2)$, it suffices to prove its closure in that space seen that the latter is separable and Hilbertian, while the equivalence between the completeness and the closure of systems in such spaces is a well known result (see e.g. [13], theorem 4, paragraph 4, chapter 3).

Let $\tilde{\Theta}_{\hat{\alpha}}$ be an element satisfying $(c1)$. One can suppose without contradicting any hypothesis that it satisfies $(c2)$ as well. Suppose next that this element is orthogonal to the full set $\{R_{\hat{\alpha}}\Theta_\alpha\}$. Therefore, by the already proved $(r1)$ and using $(c3)$ and $(c2)$, one obtains that the unique possibly non-vanishing coordinate of $L_{\hat{\alpha}}\tilde{\Theta}_{\hat{\alpha}}$ is that in the ath place. Clearly, this coordinate reads $u(a)\tilde{\Theta}_{\hat{\alpha}}(a+1)$. However, as is easily seen, up to a multiplication by a constant, this is exactly the structure of the semi-infinite vector $L_{\hat{\alpha}}\tilde{\Theta}^0_{\hat{\alpha}}$. In other words, $\tilde{\Theta}^0_{\hat{\alpha}}$ is essentially the unique element orthogonal to the whole set $\{R_{\hat{\alpha}}\Theta_\alpha\}$, which proves $(r2)$ and the proposition is completely proved. □

The adjointness such as that in equation (9) was intensively used in [12, 21] to give simplest proofs of the orthogonality relations for most of the classical polynomials (including the Askey–Wilson class). In those works, the role of $L_{\hat{\alpha}}$ and $R_{\hat{\alpha}}$ were played by the usual difference relations (considered there as the starting point) lowering and raising the degree of the polynomials and 'perturbing' the parameters. Thus, ρ^2 and $\tilde{\rho}^2$ differed only by a 'perturbation' of parameters.

Here, ρ^2 and $\tilde{\rho}^2$ will differ more than in the unique shape: we expect $\tilde{\rho}^2$ to be 'non-classical' for given 'classical' ρ^2. This will be done efficiently on the special case of Meixner polynomials $M_n^{(2,c)}(x+1)$.

To treat the question of orthogonality for the new functions $\tilde{\Psi}_{\hat{\alpha}}$, $\tilde{\Psi}_\alpha$, $\alpha \in N_\alpha$, in $\tilde{\ell}^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$, one can adopt the following method. Firstly, certify, using the reasoning of the preceding proposition, the orthogonality of $\tilde{\Psi}_{\hat{\alpha}}$ and every $\tilde{\Psi}_\alpha$, $\alpha \in N_\alpha$ (provided that $\tilde{a} = a+1; b = \tilde{b} = +\infty$). Secondly, deduce the orthogonality of $\tilde{\Psi}_\alpha$ and $\tilde{\Psi}_\beta$ for $\alpha \neq \beta$, $\alpha, \beta \in N_\alpha$, as that of eigenfunctions corresponding to mutually different eigenvalues for a symmetric operator $\tilde{H}$, provided that (obtained using simple summation by parts)

$$u(x)\tilde{\rho}^2(x)\big(\tilde{\Psi}_\alpha\Delta\tilde{\Psi}_\beta - \tilde{\Psi}_\beta\Delta\tilde{\Psi}_\alpha\big)\big|_{\tilde{a}-1}^{\tilde{b}} = 0 \tag{13}$$

where $\Delta = \boldsymbol{E}_x - 1$. It is this method that will be followed later to verify the orthogonality relations for the functions (polynomials) modifying the already evoked special case of Meixner polynomials $M_n^{(2,c)}(x+1)$.

Thus, from a Sturm–Liouville operator for which the eigenfunction expansion is known, we can generate a new (in the sense that the operators must belong to different families) Sturm–Liouville operator and the corresponding eigenfunction expansion.

However, as we are 'transforming' polynomials, the situation is that the '*transformation function*' $\Psi_{\hat{\alpha}}$ cannot be generally taken as a polynomial, while the '*transformed functions*', $\tilde{\Psi}_{\hat{\alpha}}$, $\tilde{\Psi}_\alpha$, $\alpha \in N_\alpha$ need to be polynomials or rational functions satisfying additional constraints. So, for a given Sturm–Liouville difference operator, a copious reservoir of 'good' transformation functions cannot be expected.

Remark that, in what precedes, the transformation function was deliberately chosen outside the '*transformable system*' $\Psi_\alpha, \alpha \in N_\alpha$. Supposing that the transformable functions $\Psi_\alpha(x)$ are explicit functions not only of x, but also of α, it also makes a sense to choose as the transformation function a function Ψ_γ for which $\gamma \in N_\alpha$. In that case, the new functions need to be searched for:

$$\tilde{\Psi}_\alpha = \frac{R_\gamma\Psi_\alpha}{\lambda_\alpha - \lambda_\gamma} \tag{14}$$

so the transformed $\tilde{\Psi}_\gamma$ must be understood as a limit (Hospital's rule) of the right-hand side of equation (14) for $\alpha \longrightarrow \gamma$. In orthogonal polynomial theory, transformations as in equation (14) (we mean here when the acting functions are explicit functions of the independent variable α) are referred to as the *Christoffel transformations* having as inverse the so-called *Geronimus transformations*. Many interesting studies and applications of these transformations can be found in [8, 28] and references therein.

The second-order difference operator in which we are interested here is the hypergeometric difference operator on linear lattices [22]

$$H = \sigma\Delta\nabla + \tau\Delta = (\sigma+\tau)\boldsymbol{E}_x - (2\sigma+\tau) + \sigma\boldsymbol{E}_x^{-1} \tag{15}$$

σ and τ being polynomials in x of second (at most) and first degree respectively.

Let $\lambda_n = n\tau' + \frac{1}{2}\sigma'' n(n-1)$ and $P_n(x)$, $n = 0, 1, 2, \ldots$, be its eigenvalues and the corresponding polynomial eigenfunctions (see for e.g. [22]). According to the above discussions, we can transform the polynomials as

$$\tilde{P}_n(x) = P_n(x) + f_{\hat{\alpha}}(x)P_n(x-1) = (1 + f_{\tilde{\alpha}}(x)\boldsymbol{E}_x^{-1})P_n(x) \tag{16}$$

where

$$f_{\hat{\alpha}}(x) = -\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)} \tag{17}$$

provided that the function $\Psi_{\hat{\alpha}}(x)$ does not belong to the sequence $P_n(x)$, $n = 0, 1, 2, \ldots$. Moreover, since the hypergeometric polynomials on linear lattices are (if taken in their canonical forms) polynomials not only in x but also in n (called dual polynomials), we can also use the Christoffel transformation (we here shifted x comparatively to equation (14), in order that the obtained polynomials (in n) be of degree exactly equal to x):

$$\tilde{P}_x^{(\gamma)}(n) = \frac{P_n(x+1) - \frac{P_\gamma(x+1)}{P_\gamma(x)}P_n(x)}{\lambda_n - \lambda_\gamma}. \tag{18}$$

Thus by equation (18), one can transform the polynomials dual to the hypergeometric polynomials (recall that for the Charlier, Meixner and Kravchuk polynomials, there exists self-duality) into non-classical polynomials with rational coefficients in the three-term recurrence relation (in [8], one can find detailed discussions concerning the applications of the Christoffel transformation to any family of orthogonal polynomials admitting a dual sequence of orthogonal polynomials, i.e. q-Racah polynomials and specializations (in (18), the situation is similar to that of [8], the variables x and n having interchanged their roles)).

Here, the transformation interesting us is that given by equations (16) and (17). In this case, for reasons already explained, the transformation function $\Psi_{\hat{\alpha}}(x)$ would lie outside the set of transformable polynomials $P_n(x)$. Moreover, as can be easily seen, if we expect that the transformed polynomials will be of different degree (and so be independent), we must avoid the choice of $\Psi_{\hat{\alpha}}(x)$ as a polynomial. However, for evident reasons, the ratio $\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)}$ must be a rational function, say $\frac{N_{\hat{\alpha}}(x)}{D_{\hat{\alpha}}(x)}$. Thus, if for example the new ground state (provided it belongs to $\tilde{\ell}^2(\tilde{a}, \tilde{b}; \tilde{\rho}^2)$) appears in the form $\tilde{\Psi}_{\hat{\alpha}(x)} = \frac{\hat{N}_0(x)}{D_{\hat{\alpha}}(x)}$, the new polynomials should be checked inside the set constituted by $\hat{N}_0(x)$ and the different numerators of the fractions

$$\tilde{P}_n(x) = P_n(x) - \frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)}P_n(x-1) \qquad n = 0, 1, \ldots \tag{19}$$

where

$$\frac{\Psi_{\hat{\alpha}}(x)}{\Psi_{\hat{\alpha}}(x-1)} = \frac{N_{\hat{\alpha}}(x)}{D_{\hat{\alpha}}(x)} \tag{20}$$

while the actual new weight should read $D_{\hat{\alpha}}^{-2}(x)\tilde{\rho}^2(x)$. The work of Samsonov–Ovcharov (see [26]), on the 'modification' of the Hermite polynomials, is a typical example (unique in the literature, to our best knowledge) of a realization of this scheme. For the polynomials on a lattice, this scheme will be realized below only for a special case of the Meixner polynomials, namely the $M_n^{(2,c)}(x+1)$, the question remaining open for higher polynomials (i.e. Hahn, Askey–Wilson, …).

The last remark, before starting with the calculation, concerns the degrees of the polynomials (in x) to be obtained by equations (19), (20). Writing the operator $\tilde{H}[\frac{1}{D_{\hat{\alpha}}(x)}]$ in the form

$$\tilde{\sigma}\Delta\nabla + \tilde{\tau}\Delta + \tilde{\nu} \tag{21}$$

one easily notes that the functions $\tilde{\sigma}$, $\tilde{\tau}$ and $\tilde{\nu}$ are not necessarily of degrees two, one and zero respectively. In other words, $\tilde{H}[\frac{1}{D_{\hat{\alpha}}(x)}]$ is not necessarily of hypergeometric type. Consequently, as in the case of the Hermite polynomials 'modified' by Samsonov–Ovcharov (see also the polynomials generated in [7] by 'quasi-periodicity' methods), we cannot expect that the corresponding polynomials ($\hat{N}_0(x)$ and numerators in equation (19)) will be of degrees exactly equal $n = 0, 1, 2, 3, \ldots$. There is clearly no need to be worried about this, seeing that the polynomials conserve most other properties common to usual orthogonal polynomials (completeness, orthogonality, difference and recurrence relations, difference (eigenvalue) equations, duality, …). The question of global study of orthogonal polynomials of such a category have been already raised in [8].

3. The special Meixner $M_n^{(2,c)}(x+1)$ case

Recall that the Meixner polynomials $M_n^{(\delta,c)}(x)$ are a special case of general polynomial eigenfunctions of the hypergeometric difference operator in equation (15) (see e.g. [22]). The polynomials to be transformed here are Meixner's $\boldsymbol{M_n^{(2,c)}(x+1)}$. They satisfy the second-order eigenvalue difference equation

$$[u(x)\boldsymbol{E}_x + v(x) + w(x)\boldsymbol{E}_x^{-1}]M_n^{(2,c)}(x+1) = (c-1)nM_n^{(2,c)}(x+1) \tag{22}$$

where

$$u(x) = c(x+3) \qquad v(x) = -[(c+1)x + 3c + 1] \qquad w(x) = x+1 \tag{23}$$

as well as the usual recurrence relations

$$P_{n+1} + b_n P_n + a_n^2 P_{n-1} = xP_n \tag{24}$$

with

$$b_n = -\frac{(c+1)n + 3c - 1}{c-1} \qquad a_n^2 = \frac{n(n+1)c}{(c-1)^2} \tag{25}$$

and the difference relations

$$\begin{aligned} (c-1)a_{n+1}M_{n+1}^{(2,c)}(x+1) &= [u(x)\boldsymbol{E}_x + k(x;n)]M_n^{(2,c)}(x+1) \\ -(c-1)a_n M_{n-1}^{(2,c)}(x+1) &= [u(x)\boldsymbol{E}_x + l(x;n)]M_n^{(2,c)}(x+1) \end{aligned} \tag{26}$$

where

$$k(x;n) = -x + n - 1 \qquad l(x;n) = -c(x+3+n). \tag{27}$$

They are orthogonal on $(-1, +\infty)$ with weight $\rho^2(x) = c^x(x+2)$.

The transformation functions are chosen as $c^{-x}\hat{M}_n^{(c)}(x)$, eigenfunctions of the same operator as the Meixner $M_n^{(2,c)}(x+1)$, but with eigenvalues $-(c-1)(n+2)$. Precisely, $\hat{M}_n^{(c)}(x) = M_n^{(2,c)}(-x-3)$.

The transformed rational type functions are $\tilde{\Psi}_{\gamma,0}(x)$, $\tilde{\Psi}_{\gamma,n+1}(x)$, $n, \gamma = 0, 1, 2, \ldots$

$$\begin{aligned}&\left[c(x+3)\boldsymbol{E}_x + g_\gamma(x)\right]\tilde{\Psi}_{\gamma,0}(x) = 0\\ &\tilde{\Psi}_{\gamma,n+1}(x) = \left[1 + f_\gamma(x)\boldsymbol{E}_x^{-1}\right]M_n^{(2,c)}(x+1)\end{aligned} \tag{28}$$

where

$$f_\gamma(x) = -\frac{\hat{M}_\gamma^{(c)}(x)}{c\hat{M}_\gamma^{(c)}(x-1)} \qquad g_\gamma(x) = v(x) - u(x)f_\gamma(x+1) + (c-1)(\gamma+2).$$

They satisfy the second-order eigenvalue difference equation

$$\begin{aligned}&\left[u(x)\boldsymbol{E}_x + \tilde{v}_\gamma(x) + \tilde{w}_\gamma(x)\boldsymbol{E}_x^{-1}\right]\tilde{\Psi}_{\gamma,0}(x) = -(c-1)(\gamma+2)\tilde{\Psi}_{\gamma,0}(x)\\ &\left[u(x)\boldsymbol{E}_x + \tilde{v}_\gamma(x) + \tilde{w}_\gamma(x)\boldsymbol{E}_x^{-1}\right]\tilde{\Psi}_{\gamma,n+1}(x) = (c-1)n\tilde{\Psi}_{\gamma,n+1}(x) \qquad n, \gamma = 0, 1, 2, \ldots\end{aligned} \tag{29}$$

where

$$\begin{aligned}\tilde{v}_\gamma(x) &= v(x) + f_\gamma(x)u(x-1) - u(x)f_\gamma(x+1)\\ \tilde{w}_\gamma(x) &= f_\gamma(x)\frac{w(x-1)}{f_\gamma(x-1)}.\end{aligned} \tag{30}$$

The case $\gamma = 0$ leads to classical polynomials. The cases $\gamma \geqslant 1$ lead to orthogonal rational functions with non-classical weights and from which one can extract non-classical orthogonal polynomials. Below, the computations, results and various discussions are given for the cases of $\gamma = 1, 2$, and 3.

3.1. Second-order difference eigenvalue equations, eigenfunctions

In this subsection, in each case, are written explicitly the coefficients in the second-order difference eigenvalue equations and the first four eigenfunctions.

The case $\gamma = 1$. The second-order difference eigenvalue equation reads as in equation (29) with $u(x)$ from equation (23),

$$\begin{aligned}\tilde{v}_1(x) = {} & \{-((c+1)(c-1)^2x^3 + (c-1)(4c^2-7c-5)x^2 + (6-20c^2+19c+3c^3)x\\ & +16c-4c^2)\}/\{((c-1)x-2)((c-1)x+c-3)\}\\ \tilde{w}_1(x) = {} & \frac{((c-1)x+c-3)x((c-1)x-1-c)}{((c-1)x-2)^2}.\end{aligned}$$

The new eigenfunctions are

$$\begin{aligned}\tilde{\Psi}_{1,0}(x) &= \frac{1}{(x+2)(x+1)((c-1)x-2)}\\ \tilde{\Psi}_{1,n}(x) &= \frac{\mathcal{P}_{1,n}(x)}{\mathcal{Q}_1(x)} \qquad n = 1, 2, 3, \ldots\end{aligned}$$

where $\mathcal{Q}_1(x) = (c-1)x-2$ and $\mathcal{P}_{1,n}(x)$, $n = 1, 2, 3, \ldots$, are some polynomials of degree $1, 2, 3, \ldots$ respectively.

The case $\gamma = 2$. The second-order difference eigenvalue equation reads as in equation (29) with $u(x)$ from equation (23),

$$\tilde{v}_2(x) = \{-((c+1)(c-1)^4x^5 + 3(c^2-5c-4)(c-1)^3x^4 - (c^3+41c^2-85c-53)(c-1)^2x^3 - 3(c-1) \times (c^4-c^3-63c^2+77c+34)x^2 + (72-348c^2+312c+24c^4+12c^3)x + 180c - 72c^2)\}/\{((c-1)^2x^2-(c+5)(c-1)x+6)((c-1)^2x^2 + (c-1)(c-7)x-6c+12)\}$$

$$\tilde{w}_2(x) = \{((c-1)^2x^2+(c-1)(c-7)x-6c+12)x((c-1)^2x^2 - 3(c-1)(c+1)x+2c+2c^2+2)\}/\{((c-1)^2x^2-(c+5)(c-1)x+6)^2\}.$$

The new eigenfunctions are

$$\tilde{\Psi}_{2,0}(x) = \{1\}/\{(x+2)(x+1)(2(c-1)^2x - c^2 + ((c^2+10c+1)(c-1)^2)^{1/2} - 4c+5) \times (2(c-1)^2x - c^2 - ((c^2+10c+1)(c-1)^2)^{1/2} - 4c+5)\}$$

$$\tilde{\Psi}_{2,n}(x) = \frac{\mathcal{P}_{2,n}(x)}{Q_2(x)} \qquad n = 1, 2, 3, \ldots$$

where $Q_2(x) = (c-1)^2x^2 - (c+5)(c-1)x + 6$ and $\mathcal{P}_{2,n}(x), n = 1, 2, 3, \ldots,$ are some polynomials of degree $2, 3, 4, \ldots$ respectively.

The case $\gamma = 3$. In order to avoid a fourth-order algebraic equation, we have fixed the parameter c taking $c = 1/2$.

The second-order difference eigenvalue equation reads as in equation (29) with $u(x)$ from equation (23),

$$\tilde{v}_3(x) = \{-3(x+3)(37\,536x + 709x^4 + 56\,21x^3 + 21\,946x^2 + 43x^5 + x^6 + 18\,432)\} / \{2(x+6)(x^2+15x+32)(x+7)(x^2+17x+48)\}$$

$$\tilde{w}_3(x) = \frac{(x+7)(x^2+17x+48)x(x+5)(x^2+13x+18)}{(x+6)^2(x^2+15x+32)^2}.$$

The new eigenfunctions are

$$\tilde{\Psi}_{3,0}(x) = \frac{1}{(x+1)(x+2)(x+6)(2x+15-97^{1/2})(2x+15+97^{1/2})}$$

$$\tilde{\Psi}_{3,n}(x) = \frac{\mathcal{P}_{3,n}(x)}{Q_3(x)} \qquad n = 1, 2, 3, \ldots$$

where $Q_3(x) = (x+6)(x^2+15x+32)$ and $\mathcal{P}_{3,n}(x), n = 1, 2, 3, \ldots,$ are some polynomials of degree $3, 4, 5, \ldots$ respectively.

3.2. *Orthogonality, weight functions*

We refer literally to the results of the discussions in the first section, particularly those related to the proposition 2.1. Here, $a = -1$, and the interval of orthogonality is $(0, +\infty)$. On the other hand, various solutions of equation (6) for $\tilde{\rho}_\gamma$ are

$$\tilde{\rho}_1^2(x) = \frac{\tilde{\rho}_1^2(0)(3-c)(cx-x-2)(x+2)(x+1)c^x}{4(cx+c-x-3)}$$

$$\tilde{\rho}_2^2(x) = \{c^x(x+1)(x+2)(2xc^2 - 4cx + 2x + 5 - 4c - c^2 + ((c^2+10c+1)(c-1)^2)^{1/2})$$

$$\begin{aligned}
&\times(2xc^2-4cx+2x+5-4c-c^2-((c^2+10c+1)(c-1)^2)^{1/2})\\
&\times(c^2-8c+7-((c^2+10c+1)(c-1)^2)^{1/2})\\
&\times(c^2-8c+7+((c^2+10c+1)(c-1)^2)^{1/2})\tilde{\rho}_2^2(0)\}\\
&/\{2(-5+4c+c^2-((c^2+10c+1)(c-1)^2)^{1/2})\\
&\times(-5+4c+c^2+((c^2+10c+1)(c-1)^2)^{1/2})\\
&\times(2xc^2-4cx+2x+c^2+7-8c+((c^2+10c+1)(c-1)^2)^{1/2})\\
&\times(2xc^2-4cx+2x-((c^2+10c+1)(c-1)^2)^{1/2}+c^2+7-8c)\}
\end{aligned}$$

$$\begin{aligned}
\tilde{\rho}_3^2(x)=\{&2^{-x}7(x+1)(x+2)(x+6)(2x+15-97^{1/2})(2x+15+97^{1/2})(-17+97^{1/2})\\
&\times(17+97^{1/2})\tilde{\rho}_3^2(0)\}/\{12(x+7)(2x-97^{1/2}+17)(2x+17+97^{1/2})\\
&\times(15+97^{1/2})(-15+97^{1/2})\}.
\end{aligned}$$

Recalling that the bottom function $\tilde{\Psi}_{\gamma,0}(x)$ is the solution of the equation (see equation (10))

$$L_{\tilde{\alpha}}\tilde{Y}(x)=0 \tag{31}$$

one directly deduces from proposition 2.1, the orthogonality, on the interval $(0,+\infty)$, of the bottom function $\tilde{\Psi}_{\gamma,0}(x)$ with each of the elements from the higher ladder $\tilde{\Psi}_{\gamma,n}(x)$, $n=1,2,\ldots$, relative to the weights $\tilde{\rho}_\gamma^2(x)$. On the other hand, as one can easily show, equation (13) is verified for $\alpha,\beta=1,2,\ldots,\alpha\neq\beta$. Consequently, the transformed operator $\tilde{H}_\gamma$ (in the left-hand side of equation (29)) is symmetric in the subspace of $\ell^2(0,+\infty;\tilde{\rho}_\gamma^2(x))$ generated by the higher ladder $\tilde{\Psi}_{\gamma,n}(x)$, $n=1,2,\ldots$. Hence, the functions from this ladder are mutually orthogonal there. Thus, we have obtained that all the new functions $\tilde{\Psi}_{\gamma,n}(x)$, $n=0,1,2,\ldots$, are mutually orthogonal on the interval $(0,+\infty)$ relatively to the weights $\tilde{\rho}_\gamma^2(x)$ given above.

A direct consequence of this, are the orthogonality relations between polynomials related to the $\tilde{\Psi}$-functions. Namely, the following.

The polynomials

$$\begin{aligned}
&\bar{\mathcal{P}}_{1,0}(x)=1\\
&\bar{\mathcal{P}}_{1,n}(x)=(x+2)(x+1)\mathcal{P}_{1,n}(x)\qquad n=1,2,3,\ldots
\end{aligned}$$

are orthogonal on the same interval as $\tilde{\Psi}_{1,n}(x)$ but now with the weight

$$\tilde{\varrho}_1^2(x)=\big[(x+1)(x+2)Q_1(x)\big]^{-2}\tilde{\rho}_1^2(x).$$

Identically, the polynomials

$$\begin{aligned}
&\bar{\mathcal{P}}_{2,0}(x)=Q_2(x)\\
&\bar{\mathcal{P}}_{2,n}(x)=[\tilde{\Psi}_{2,0}(x)]^{-1}\mathcal{P}_{2,n}(x)\qquad n=1,2,3,\ldots
\end{aligned}$$

are orthogonal on the same interval as $\tilde{\Psi}_{2,n}(x)$ but now with the weight

$$\tilde{\varrho}_2^2(x)=\big[[\tilde{\Psi}_{2,0}(x)]^{-1}Q_2(x)\big]^{-2}\tilde{\rho}_2^2(x).$$

Finally, the polynomials

$$\begin{aligned}
&\bar{\mathcal{P}}_{3,0}(x)=(x+6)^{-1}Q_3(x)\\
&\bar{\mathcal{P}}_{3,n}(x)=[(x+6)\tilde{\Psi}_{3,0}(x)]^{-1}\mathcal{P}_{3,n}(x)\qquad n=1,2,3,\ldots
\end{aligned}$$

are orthogonal on the same interval as $\tilde{\Psi}_{3,n}(x)$ but now with the weight

$$\tilde{\varrho}_3^2(x)=\big[[(x+6)\tilde{\Psi}_{3,0}(x)]^{-1}Q_3(x)\big]^{-2}\tilde{\rho}_3^2(x).$$

3.3. Completion

The completion of the functions $\tilde{\Psi}_{\gamma,n}(x)$, $n = 0, 1, 2, \ldots$, in $\tilde{\ell}^2(0, +\infty; \tilde{\rho}_\gamma^2(x))$ follows from that of the Meixner $M_n^{(2,c)}(x+1)$ in $\ell^2(-1, +\infty; \rho^2(x))$. This is a direct consequence of proposition 2.1. In particular here, $a = -1$, so that $\tilde{\Theta}_{\hat{\alpha}}$ and Θ_α considered in proposition 2.1 are obtained respectively from $\tilde{\Psi}_{\gamma,0}(x)$ and $M_\alpha^{(2,c)}(x+1)$ by sending coordinates corresponding to negative x to zero.

On the other hand, the completion of the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x), n = 0, 1, 2\ldots$, in $\tilde{\ell}^2(0, +\infty; \tilde{\varrho}_\gamma^2(x))$ follows from that of $\tilde{\Psi}_{\gamma,n}(x)$, $n = 0, 1, 2\ldots$, in $\tilde{\ell}^2(0, +\infty; \tilde{\rho}_\gamma^2(x))$.

This completion holds well in spite of the fact that, in each of the constructed sequences of polynomials, there exists at least one number $n \in Z^+$ such that no polynomial from that sequence has degree exactly equal to n.

3.4. Difference and recurrence relations

As other deducible properties of the new functions, one finds from equation (28) and the recurrence relations for the Meixner $M_n^{(2,c)}(x+1)$ (see equation (25)) polynomials, the following five-term recurrence relations:

$$\begin{aligned}
&\tilde{\Psi}_{\gamma,n+2}(x) + [b_{n+1} + b_n - 2x + 1]\tilde{\Psi}_{\gamma,n+1}(x) + [a_{n+1}^2 + a_n^2 + (b_n - x + 1)(b_n - x)]\tilde{\Psi}_{\gamma,n}(x) \\
&\qquad + [a_n^2(b_n + b_{n-1} - 2x + 1)]\tilde{\Psi}_{\gamma,n-1}(x) + a_n^2 a_{n-1}^2 \tilde{\Psi}_{\gamma,n-2}(x) = 0 \\
&\qquad n = 3, 4, \ldots.
\end{aligned}$$

However, as easily seen, these recurrence relations are satisfied by any linear combination (with coefficients depending on x and not on n) of Meixner $M_n^{(2,c)}(x+1)$, and $M_n^{(2,c)}(x)$. In other words, the relations do not depend of $f_\gamma(x)$.

More characteristic recurrence relations for these functions are the following three-term recurrence relations that one can establish from equation (28), the difference eigenvalue equation (22) and the difference relations equations (26), (27) satisfied by the Meixner $M_n^{(2,c)}(x+1)$ polynomials:

$$\begin{aligned}
&[(c-1)a_{n+1}(f_\gamma(x)l(x-1;n) - w(x) - f_\gamma(x)l(x;n) + f_\gamma(x)(v(x) - \lambda_n) \\
&\qquad - f_\gamma^2(x)u(x-1))]\tilde{\Psi}_{\gamma,n+1}(x) + [f_\gamma(x)k(x-1;n)l(x;n) \\
&\qquad - f_\gamma(x)k(x-1;n)(v(x) - \lambda_n) + f_\gamma^2(x)k(x-1;n)u(x-1) - w(x)l(x;n) \\
&\qquad - f_\gamma(x)k(x;n)l(x-1;n) + k(x;n)w(x) + f_\gamma(x)l(x-1;n)(v(x) - \lambda_n) \\
&\qquad - f_\gamma^2(x)u(x-1)l(x-1;n)]\tilde{\Psi}_{\gamma,n}(x) + [(c-1)a_n(f_\gamma(x)(v(x) - \lambda_n) \\
&\qquad - f_\gamma(x)k(x;n) - f_\gamma^2(x)u(x-1) + f_\gamma(x)k(x-1;n) - w(x))]\tilde{\Psi}_{\gamma,n-1}(x) = 0 \\
&\qquad n = 2, 3, \ldots.
\end{aligned}$$

The obtained functions satisfy also the following third-order difference relations, which follow clearly from the difference relations equations (26) and (27) satisfied by the Meixner $M_n^{(2,c)}(x+1)$ polynomials, together with equation (28) and its inverse:

$$\begin{aligned}
-\lambda_\gamma(c-1)a_{n+1}\tilde{\Psi}_{\gamma,n+1}(x) &= \{[1 + f_\gamma(x)\boldsymbol{E}_x^{-1}][u(x)\boldsymbol{E}_x + k(x;n)] \\
&\qquad \times [u(x)\boldsymbol{E}_x + g_\gamma(x)]\}\tilde{\Psi}_{\gamma,n}(x) \\
\lambda_\gamma(c-1)a_n\tilde{\Psi}_{\gamma,n-1}(x) &= \{[1 + f_\gamma(x)\boldsymbol{E}_x^{-1}] \\
&\qquad \times [u(x)\boldsymbol{E}_x + l(x;n)][u(x)\boldsymbol{E}_x + g_\gamma(x)]\}\tilde{\Psi}_{\gamma,n}(x) \quad n = 2, 3, \ldots.
\end{aligned}$$

Using the second-order difference eigenvalue equation (29) satisfied by the new functions and the preceding relations, one can reach, if necessary, the following first-order difference relations:

$$-\lambda_\gamma(c-1)a_{n+1}\tilde{\Psi}_{\gamma,n+1}(x) = \{[-u(x)(v(x+1)-\lambda_n) + u(x)g_\gamma(x+1) + k(x;n)u(x) + f_\gamma(x)u(x-1)u(x) - u(x)f_\gamma(x)k(x-1;n)g_\gamma(x-1)/w(x)]\boldsymbol{E}_x + [-u(x)w(x+1) + k(x;n)g_\gamma(x) + f_\gamma(x)g_\gamma(x)u(x-1) + f_\gamma(x)k(x-1;n)u(x-1) - f_\gamma(x)g_\gamma(x-1)k(x-1;n)(v(x)-\lambda_n)/w(x)]\} \times\tilde{\Psi}_{\gamma,n}(x)$$

$$\lambda_\gamma(c-1)a_n\tilde{\Psi}_{\gamma,n-1}(x) = \{[-u(x)(v(x+1)-\lambda_n) + u(x)g_\gamma(x+1) + l(x;n)u(x) + f_\gamma(x)u(x-1)u(x) - u(x)f_\gamma(x)l(x-1;n)g_\gamma(x-1)/w(x)]\boldsymbol{E}_x + [-u(x)w(x+1) + l(x;n)g_\gamma(x) + f_\gamma(x)g_\gamma(x)u(x-1) + f_\gamma(x)l(x-1;n)u(x-1) - f_\gamma(x)g_\gamma(x-1)l(x-1;n)(v(x)-\lambda_n)/w(x)]\} \times\tilde{\Psi}_{\gamma,n}(x) \qquad n = 2, 3, \ldots.$$

It is clear that the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x)$ satisfy the same recurrence relations as the functions $\tilde{\Psi}_{\gamma,n}(x)$. Their difference relations are also obviously deduced from those of $\tilde{\Psi}_{\gamma,n}(x)$.

3.5. Duality

Parallel to the above results for the functions $\tilde{\Psi}_{\gamma,n}(x)$ and the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x)$, one can of course deduce the corresponding results for the polynomials dual to the polynomials $\bar{\mathcal{P}}_{\gamma,n}(x)$ (more precisely, dual to the functions $\tilde{\Psi}_{\gamma,n}(x)$), $n = 1, 2, \ldots$, thanks to the existence of dual (self-dual) polynomials for the Meixner polynomials. We will denote $\mathcal{D}_{\gamma,x}(n) = \breve{\rho}(x)\tilde{\Psi}_{\gamma,n+1}(x)$, $n,\ x = 0, 1, 2, \ldots$, $\breve{\rho}(x) = c^x\Gamma(x+3)$. $\mathcal{D}_{\gamma,x}(n)$ are polynomials in n, of degree exactly equal $x+1$ and satisfy the usual three-term recurrence relations (in x)

$$P_{x+1} + b_{\gamma,x}P_x + a^2_{\gamma,x}P_{x-1} = nP_x \tag{32}$$

with (see equation (29))

$$b_{\gamma,x} = \tilde{v}_\gamma(x) \qquad a^2_{\gamma,x} = \tilde{w}_\gamma(x)u(x-1)/(c-1)^2. \tag{33}$$

The polynomials in n

$$\check{\mathcal{D}}_{\gamma,0} = 1 \qquad \check{\mathcal{D}}_{\gamma,1} = (c-1)n - b_{\gamma,0} \tag{34}$$

$$\check{\mathcal{D}}_{\gamma,x+1} = ((c-1)n - b_{\gamma,x})\check{\mathcal{D}}_{\gamma,x} - a^2_{\gamma,x}\check{\mathcal{D}}_{\gamma,x-1} \qquad x = 1, 2, \ldots \tag{35}$$

are for $x = 0, 1, 2\ldots$ of degree exactly equal to x and are clearly orthogonal, the condition required in the well known Favard theorem ($b_{\gamma,x}$ real and $a^2_{\gamma,x} > 0,\ x \geqslant 1$) being satisfied. On the other hand, $\mathcal{D}_{\gamma,x}(n) = \check{\mathcal{D}}_{\gamma,x}(n)\breve{\rho}(0)\tilde{\Psi}_{\gamma,n+1}(0)$, where from equation (28) $\tilde{\Psi}_{\gamma,n+1}(0) = (\gamma)_n(\frac{c}{c-1})^n[f_\gamma(0) + 1 + \frac{1}{2}n(1-\frac{1}{c})]$, $(\gamma)_0 = 1$, $(\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1)$, $n = 1, 2\ldots.$ Consequently,

$$\tilde{\Psi}_{\gamma,n+1}(x) = \frac{(\gamma)_n(\frac{c}{c-1})^n}{c^x\Gamma(x+3)}\cdot\left[2f_\gamma(0) + 2 + n\left(1-\frac{1}{c}\right)\right]\check{\mathcal{D}}_{\gamma,x}(n) \qquad x, n = 0, 1, \ldots. \tag{36}$$

4. Concluding remarks and outlook

Concluding this paper, let us first remark that the obtained polynomials and functions can, in their turn, be transformed in a similar manner. Here, the reservoir of transformation functions is composed of

$$\Phi_{\hat{\alpha},n}(x) = [1 + f_{\hat{\alpha}}\boldsymbol{E}_x^{-1}]c^{-x}\hat{M}_n(x) \qquad n, \hat{\alpha} \in \mathcal{N} \qquad n \neq \hat{\alpha} \tag{37}$$

and, clearly, the process can, in principle, be repeated infinitely many times.

The existence of other efficiently transformable particular cases of the hypergeometric difference operator H in equation (15), 'higher' than the special Meixner $M_n^{(2,c)}(x+1)$ case treated here, is of course not precluded. As noted in [6], it is neither precluded that such solvable Hamiltonians could be generated by self-similarity or shape-invariance (symmetry) techniques. The fact that any two of the three methods evoked in the introduction may lead to the same result is illustrated by the already noted fact that the non-classical polynomials in [7, equation (4.13)], related to the Hermite polynomials generated there by 'quasi-periodicity' methods, have been rediscovered in [26], using the 'modification method' on the Hermite polynomials (see the case $m = 2$ in [26]).

It is worth remarking that no efficient 'modification' of a fourth-order Sturm–Liouville difference (or differential) operator (using for example one of the formulae from section 2.2.2 of the second chapter in [5]) is known in the literature.

Remark next that if the constraint of being polynomials is rejected as well for the 'transformable' functions as for the 'transformed' ones (retaining only the constraints of integrability and completeness), the situation becomes much easier. Thus, nowadays, the method used here is one of the main tools for generating new exactly solvable Hamiltonians in quantum mechanics (see e.g. [19, 23–25, 30]).

In [29], transformations of orthogonal polynomials were studied whose weights are obtained by multiplying a *rational* function by the original ones. Let us note although the weights $\tilde{\varrho}_\gamma^2(x)$, $\gamma = 1, 2, 3$ are obtained by multiplying rational functions by $\rho_\gamma^2(x)$, $\gamma = 1, 2, 3$, respectively, the situation here is generally that the new weights are obtained from the old ones by multiplying them not by rational functions but by a ratio of products of Gamma or maybe exponential functions, so the polynomials studied here are not in principle of the same kind as those from [29]. Another remarkable difference resides in the fact that the polynomials studied in [29] are 'ordinary' orthogonal polynomials satisfying a three-term recurrence relation and have degrees exactly equal to $0, 1, 2, \ldots$, while the orthogonal polynomials studied here do not satisfy the usual three-term recurrence relation and are not of degrees exactly equal to $0, 1, 2, \ldots$ (see above).

As already noted the polynomials studied here conserve most of the properties of 'ordinary' discrete orthogonal polynomials (completeness, orthogonality, difference and recurrence relations, difference (eigenvalue) equations, duality, ...). A global study of orthogonal polynomials of such a category (other explicit examples are found in [7, 8, 26]) is of a certain interest.

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Appendix

In this appendix, we give for the concrete cases of $\gamma = 1$, 2 and 3 (see tables A.1–A.3 respectively), the transformed interval of orthogonality, the transformed weight function and the first four transformed polynomials.

Table A.1. The case $\gamma = 1$.

(a, b)	$(0, +\infty)$
$\tilde{\varrho}_1^2(x)$	$\dfrac{\tilde{\varrho}_1^2(0)4(3-c)c^x}{(cx-x-2)(x+2)(x+1)(cx+c-x-3)}$
$\bar{\mathcal{P}}_{1,0}(x)$	1
$\bar{\mathcal{P}}_{1,1}(x)$	$(x+1)(x+2)((c-1)^2x-3c+3)$
$\bar{\mathcal{P}}_{1,2}(x)$	$(c-1)(x+1)(x+2)((c-1)^2x^2+3(c-1)^2x-8c)$
$\bar{\mathcal{P}}_{1,3}(x)$	$(c-1)((c-1)^3x^3+(7c-2)(c-1)^2x^2+(c-1)(12c^2-19c-3)x-30c^2)$ $(x+1)(x+2)$

Table A.2. The case $\gamma = 2$.

(a, b)	$(0, +\infty)$
$\tilde{\varrho}_2^2(x)$	$\{72c^x(((c-1)(c+5))^2-(c^2+10c+1)(c-1)^2)$ $(((c-1)(c-7))^2-(c^2+10c+1)(c-1)^2)\tilde{\varrho}_2^2(0)\}$ $/\{((2x(c-1)^2-(c-1)(c+5))^2-(c^2+10c+1)(c-1)^2)$ $((c-1)^2x^2-(c+5)(c-1)x+6)^2(x+1)(x+2)$ $((2x(c-1)^2+(c-1)(c-7))^2-(c^2+10c+1)(c-1)^2)\}$
$\bar{\mathcal{P}}_{2,0}(x)$	$(c-1)^2x^2-(c+5)(c-1)x+6$
$\bar{\mathcal{P}}_{2,1}(x)$	$(c-1)((c-1)^2x^2-(c+7)(c-1)x+12)$ $(x+1)(x+2)((2(c-1)^2x-(c+5)(c-1))^2-(c^2+10c+1)(c-1)^2)$
$\bar{\mathcal{P}}_{2,2}(x)$	$(c-1)((c-1)^3x^3+(2c-7)(c-1)^2x^2-(c-1)(3c^2+19c-12)x+30c)$ $(x+1)(x+2)((2(c-1)^2x-(c+5)(c-1))^2-(c^2+10c+1)(c-1)^2)$
$\bar{\mathcal{P}}_{2,3}(x)$	$(x+1)(x+2)((2(c-1)^2x-(c+5)(c-1))^2-(c^2+10c+1)(c-1)^2)$ $(c-1)((c-1)^4x^4+6(c-1)^4x^3+(5c^2-46c+5)(c-1)^2x^2$ $-12(c^2+7c+1)(c-1)^2x+108c^2)$

Table A.3. The case $\gamma = 3$.

(a, b)	$(0, +\infty)$
$\tilde{\varrho}_3^2(x)$	$\{2^{-x}32^4 2016\tilde{\varrho}_3^2(0)\}$ $/\{(x+1)(x+2)(x+6)((2x+15)^2-97)(x^2+15x+32)^2$ $(x+7)((2x+17)^2-97)\}$
$\bar{\mathcal{P}}_{3,0}(x)$	$x^2+15x+32$
$\bar{\mathcal{P}}_{3,1}(x)$	$-(x+1)(x+2)((2x+15)^2-97)(x+15)(x+4)(x+8)$
$\bar{\mathcal{P}}_{3,2}(x)$	$\frac{1}{2}(x+1)(x+2)((2x+15)^2-97)(x^4+24x^3+137x^2-66x-1152)$
$\bar{\mathcal{P}}_{3,3}(x)$	$-\frac{1}{4}(x+1)(x+2)((2x+15)^2-97)(x^5+18x^4-7x^3-960x^2-1740x$ $+4032)$

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